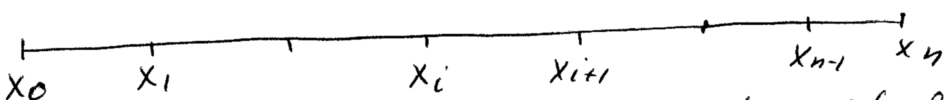
Integrals and mass

→ Let ρ be the density of a linear (1-dim) object; $\rho = \text{const}$ and ℓ be the length of the object. Then $\underline{m = \rho \cdot \ell}$ gives us the mass of the object

→ What happens if ρ is not constant?
$\rho = \rho(x)$; it depends on x, $a \le x \le b$,
$\ell = b - a$ is the length of the object.
We divide (partition) the object into n subintervals (pieces) of $\Delta x = \frac{b-a}{n}$.



$[x_i; x_{i+1}]$

Assume that the density of each piece is constant and it is equal to $\rho(x_i)$ or $\rho(x_{i+1})$
then the mass of i-th piece is $\rho(x_i) \cdot \Delta x$
The total mass of the object $= \sum_{i=0}^{n} \rho(x_i)\Delta x$.
Let $\Delta x \to 0$, then the Riemann sum $\sum_{i=0}^{n} \rho(x_i)\Delta x \to \int_a^b \rho(x)\,dx$
approaches the definite integral.

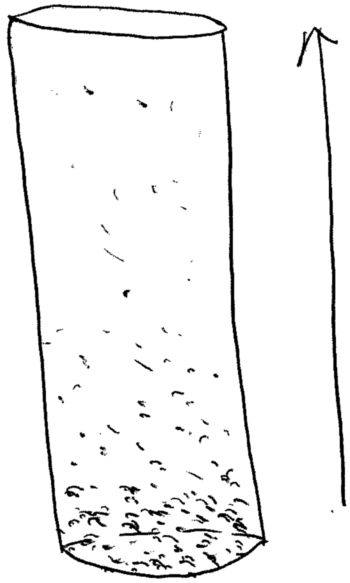
$$\boxed{\text{MASS} = \int_a^b \rho(x)\,dx}$$

$\boxed{\text{Example}}$ Imagine a cylindrical tank with phytoplankton in it. They swim and settle. The density at the bottom is higher than the density at the top. Suppose the cylinder is 10 cm high and the density $\rho(z)$ is

$$\rho(z) = e^{-0.1z} \left[\frac{g}{cm}\right]$$

→ What is the total mass of the phytoplankton in the cylinder?

→ What is the average density of the phytoplankton in the cylinder?



$z=10$ $\qquad \rho(10) = e^{-1} = \frac{1}{e}$

z

$z=0$ $\qquad \rho(0) = e^{0} = 1$

Mass $= \int_0^{10} \rho(z)\,dz = \int_0^{10} e^{-0.1z}\,dz$ = Lecture 2 guess-and check meth

$$= \frac{e^{-0.1z}}{-0.1}\Bigg|_{z=10}^{z=10} = -\frac{1}{0.1}\left[e^{-0.1\cdot 10} - e^{-0.1\cdot 0}\right] =$$

$$= -10\left[e^{-1} - 1\right] = -10\left[\frac{1}{e} - 1\right] =$$

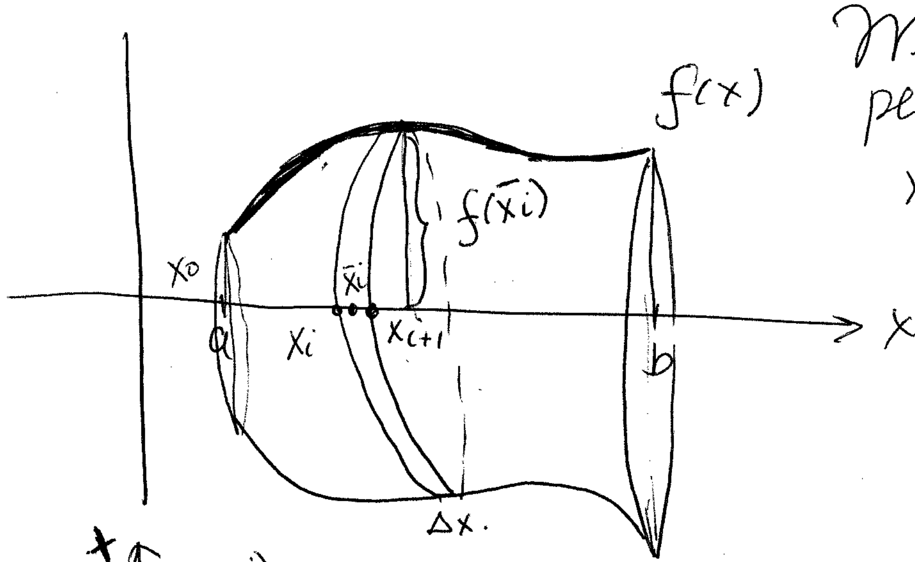
$$= 10\left[1 - \frac{1}{e}\right] \simeq 6.31 \ (g).$$

Average density $= \dfrac{\text{total mass}}{\text{hight}} = \dfrac{6.31}{10} \simeq 0.631 \left[\dfrac{g}{cm}\right]$

$$\boxed{\text{Average density} = \bar{\rho} = \frac{\int_a^b \rho(x)\,dx}{b-a}.}$$

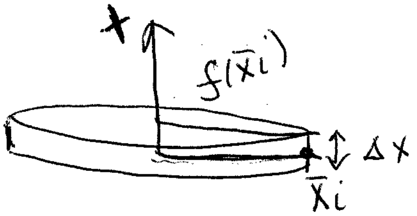
Integrals and Volumes

Imagine a positive function $f(x)$, $a \le x \le b$.
The region bounded by the curve and the x-axis between $x=a$ and $x=b$ is revolved around the x-axis. Find the volume of this solid of revolution.



We slice the region perpendicular to the x-axis, giving cylinders of height Δx

$$\Delta x = \frac{b-a}{n}.$$



$\bar{x}_i \in [x_i, x_{i+1}]$

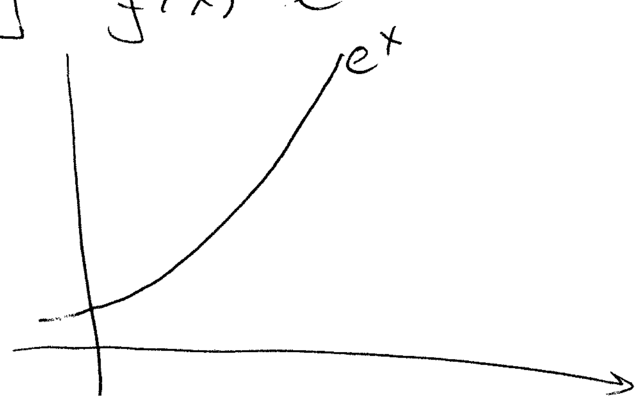
[For example, $\bar{x}_i$ can be x_i.]

Volume of i-th cylinder (of i-th slice) = Area of crossection $\times\ \Delta x$ =

$$= \text{Area of cyrcle} \times \Delta x = \pi\left[f(\bar{x}_i)\right]^2 \cdot \Delta x.$$

$$\text{Total Volume} = \sum_{i=0}^{n-1} \pi\left[f(\bar{x}_i)\right]^2 \cdot \Delta x \xrightarrow[\substack{\Delta x \to 0 \\ n \to \infty}]{} \int_a^b \pi\left(f(x)\right)^2 dx$$

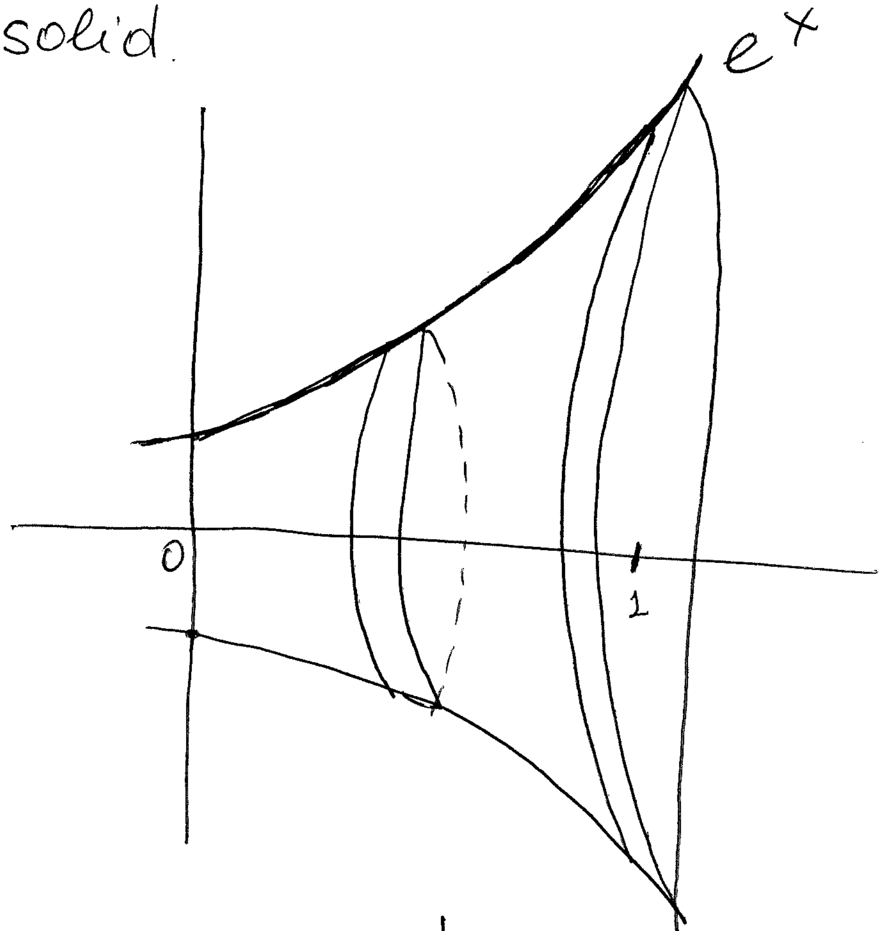
Example $f(x) = e^x$



Example $f(x) = e^x$, $0 \le x \le 1$

Imagine the graph of $f(x)$ rotating around the x-axis.

What is the volume of the resulting solid.



$$\text{Volume} = \pi \int_0^1 f^2(x)\,dx = \pi \int_0^1 e^{2x}\,dx =$$

$$= \pi \frac{e^{2x}}{2}\Big|_0^1 = \frac{\pi}{2}\left[e^2 - e^0\right] = \frac{\pi}{2}\left[e^2 - 1\right]$$

Partial Fractions

A rational function $f(x)$ can be presented as a ratio of two polynomials of x:

$$f(x) = \frac{P(x)}{Q(x)}.$$

For example: $\frac{P(x)}{Q(x)} = \frac{x^3+2x^2+7}{x+1}$ or $\frac{x+1}{x^2-3x+1}$; $\frac{1}{x^2-4}$; $\frac{1}{x+3}$ and so on.

→ We assume that $\deg(Q) \leq 2$ (the highest power of x of $Q(x)$ cannot be greater than 2)

→ Also, $\deg(P) = \{0 \text{ or } 1 \text{ or } 2 \text{ or } 3\}$. Namely, $P(x)$ can be a constant, a linear, a quadratic or a cubic polynomial.

Goal: $\int f(x)dx = \int \frac{P(x)}{Q(x)} dx$ - ?

Method: We want to decompose (split) a rational function $f(x)$ into a sum of simpler rational functions, which we know how to integrate.

$\boxed{\text{Case 1}}$ $\deg(P) < \deg(Q)$, $Q(x) = ax^2+bx+c$

$Q(x)$ is a quadratic polynomial with two distinct roots x_1 and x_2 $\boxed{D>0}$

$D = b^2-4ac$; $x_1 = \frac{-b+\sqrt{D}}{2a}$, $x_2 = \frac{-b-\sqrt{D}}{2a}$

$Q(x) = ax^2 + bx + c = a(x - x_1)(x - x_2)$.

In this case, we do the following partial fractions decomposition:

$$\frac{P(x)}{Q(x)} = \frac{A}{x - x_1} + \frac{B}{x - x_2}, \quad A, B \text{ are unknown constants.}$$

Then we find A, B:

$$\int \frac{P}{Q}\,dx = \int \frac{A}{x - x_1}\,dx + \int \frac{B}{x - x_2}\,dx$$ and

integrate the RHS.

<u>Example</u>: $\frac{P(x)}{Q(x)} = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$

Multiplying the both sides by $1-x^2$, get

$$1 = A(1+x) + B(1-x) = A + Ax + B - Bx \Rightarrow$$

$$1 = A + B + (A - B)x$$

Equating powers of x on the LHS and the RHS gives

$$A - B = 0 \quad \Rightarrow A = B$$

$$A + B = 1 \quad \Rightarrow A = 1 - B$$

$$1 - B = B$$

$$2B = 1$$

$$B = \tfrac{1}{2}, \quad A = \tfrac{1}{2}$$

Thus, $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} = \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$

$$\int \frac{dx}{1-x^2} = \int \frac{dx}{2(1-x)} + \int \frac{dx}{2(1+x)} =$$

$$= \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}\left(-\ln|1-x| + \ln|1+x|\right) + C$$

! **Remark**

$$\int \frac{dx}{1-x} = \boxed{\begin{array}{c} 1-x = u(x) \\ -dx = du \\ dx = -du \end{array}} =$$

!

$$= \int -\frac{du}{u} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|1-x| + C$$

However,

$$\int \frac{dx}{x-1} = \boxed{\begin{array}{c} x-1 = u(x) \\ dx = du \end{array}} = \int \frac{du}{u} =$$

$$= \ln|u| + C = \ln|x-1| + C.$$

no minus in this case !

Example $\int \frac{x-9}{(x+5)(x-2)}\,dx$

$$\frac{P(x)}{Q(x)} = \frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

Multiply both sides by $(x+5)(x-2)$:

$$x-9 = A(x-2) + B(x+5) = Ax - 2A + Bx + 5B \Rightarrow$$

$$x-9 = (A+B)x + (5B-2A)$$

$$\begin{cases} A+B=1 \\ 5B-2A=-9 \end{cases} \Rightarrow \quad A=2\ ,\ B=-1$$

Thus, $\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \frac{2\,dx}{x+5} - \int \frac{dx}{x-2} =$

$= 2\ln|x+5| - \ln|x-2| + C.$

More examples on partial fraction decomposition

$$\frac{3x-2}{x(x-2)} = \frac{1}{x} + \frac{2}{x-2}$$

$$\frac{2x+1}{x^2+x-2} = \frac{2x+1}{(x-1)(x+2)} = \frac{1}{x-1} + \frac{1}{x+2}$$

(Please check).

Case 2 $\quad \mathcal{D} = 0. \quad f(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{(x-x_1)^2} =$

$$= \frac{A_1}{x-x_1} + \frac{A_2}{(x-x_1)^2}.$$

Find A_1, A_2, then integrate the RHS.

Example $\quad f(x) = \frac{P(x)}{Q(x)} = \frac{x+5}{x^2-4x+4} = \frac{x+5}{(x-2)^2} =$

= Partial fraction decomposition is given by

$$\frac{A_1}{x-2} + \frac{A_2}{(x-2)^2}.$$

As before, we multiply both sides by $(x-2)^2$:

$$x+5 = A_1(x-2) + A_2 = A_1 x + (A_2 - 2A_1)$$

$$\begin{cases} A_1 = 1 \\ A_2 - 2A_1 = 5 \\ A_2 = 5 + 2A_1 = 7. \end{cases}$$

$$\frac{x+5}{(x-2)^2} = \frac{1}{x-2} + \frac{7}{(x-2)^2}$$

Thus, $\int \frac{x+5}{(x-2)^2}\,dx = \int \frac{dx}{x-2} + \int \frac{7}{(x-2)^2} =$

check and guess method

$$= \ln|x-2| + 7\left(-\frac{1}{x-2}\right) + C.$$

$$\int \frac{dx}{x^2} = -\frac{1}{x} + C.$$

Case 3 | $\mathcal{D} < 0$, i.e $Q(x)$ has no (real) roo

Method: Completing the square in $Q(x)$.

$$\frac{P(x)}{Q(x)} = \frac{1}{x^2+2x+2} \quad ; \quad \int \frac{dx}{x^2+2x+2} - ?$$

$$x^2+2x+2 = 0$$

$$\mathcal{D} = 4 - 4\cdot 2 = -4 < 0.$$

$$\frac{1}{x^2+2x+2} = \frac{1}{(x^2+2x+1)+1} = \frac{1}{(x+1)^2+1}$$

$$\int \frac{dx}{(x+1)^2+1} = \boxed{\begin{array}{l} \text{we use the} \\ \text{substitution } u(x) = x+1 \\ \qquad\qquad\qquad du = dx \end{array}} =$$

$$= \int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+1) + C.$$

$\mathcal{D} < 0.$

Example $\quad \int \frac{3x+2}{x^2-2x+5}\, dx = \int \frac{3x+2}{(x^2-2x+1)+4}\, dx =$

$$= \int \frac{3x+2}{(x-1)^2+4}\, dx = \boxed{\begin{array}{l} \text{the substitution } u(x) = x-1 \\ \qquad\qquad\qquad du = dx \\ \qquad\qquad\qquad x = u+1 \end{array}}$$

$$= \int \frac{3(u+1)+2}{u^2+4}\, du = \int \frac{3u+5}{u^2+4}\, du =$$

$$= \underbrace{3\int \frac{u\,du}{u^2+4}}_{I_1} + \underbrace{5\int \frac{du}{u^2+4}}_{I_2}. \quad ☆$$

$$I_1 = \int \frac{u\,du}{u^2+4} \qquad \boxed{\begin{array}{l} u^2+4 = z(u) \\ 2u\,du = dz \quad \text{or} \quad u\,du = \frac{dz}{2} \end{array}}$$

$$= \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\ln|z| + C = \frac{1}{2}\ln|u^2+4| + C.$$

$$I_2 = \int \frac{du}{u^2+4} = \int \frac{du}{4\left(\frac{u^2}{4}+1\right)} = \frac{1}{4}\int \frac{du}{\frac{u^2}{4}+1} =$$

$$= \frac{1}{4}\int \frac{du}{\left(\frac{u}{2}\right)^2+1} = \boxed{\begin{array}{l} \frac{u}{2} = z \\ du = 2dz \end{array}} = \frac{1}{4}\int \frac{2dz}{z^2+1} =$$

$$= \frac{1}{2}\int \frac{dz}{z^2+1} = \frac{1}{2}\operatorname{artan}(z) + C = \frac{1}{2}\operatorname{artan}\left(\frac{u}{2}\right) + C.$$

$$\frac{3}{2}\ln|u^2+4| + \frac{5}{2}\arctan\left(\frac{u}{2}\right) + C =$$

$$= \frac{3}{2}\ln\left|(x-1)^2+4\right| + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

If $\deg(P) > \deg(Q)$, then use long division first, then Case 1, Case 2 or Case 3.